数理解析研究所講究録
第 655 巻 1988 年 1-19

A distributed rewriting system in a cellular space

NISHIDA, Taishin
西田 泰伸

International Institute for Advanced Study
of Social Information Science
FUJITSU LIMITED

Introduction

In this paper we describe a developmental system, called a distributed rewriting system or a DR system for short, which generates a sequence of two dimensional patterns of strings. A two dimensional pattern of strings or a configuration is a mapping from the two dimensional rectangular array to the set of strings over a finite alphabet. A DR system consists of a set of distributed rewriting rules and an initial configuration. Distributed rewriting rules resemble to rewriting rules of a context free grammar or a 0L system. That is, distributed rewriting rules are of the form

$$a \rightarrow (b_1, d_1) \cdots (b_n, d_n),$$

where a and b_i are symbols of the alphabet and d_i is one of the direction symbols in $\{\uparrow, \downarrow, \leftarrow, \rightarrow, \cdot\}$ for $1 \leq i \leq n$. If a is contained in a string at the point (x, y) and $d_i = \rightarrow$ ($\leftarrow$, $\uparrow$, $\downarrow$, or $\cdot$), then b_i is distributed at the point $(x+1, y)$ ($(x-1, y)$, $(x, y+1)$, $(x, y-1)$, or (x, y)) for $1 \leq i \leq n$.

After giving some preliminaries, we define a DR system formally in Section 2. We will prove that for a deterministic DR system it is decidable whether the symbols derived by the system are contained in a bounded region or not (Theorem 3.8). In Section 4 we will prove that for a deterministic DR system it is decidable whether the number of symbols in each point is bounded or not (Theorem 4.8).

Since 1960's, many array and web generators have been investigated (e.g., [1], [2], and [4]). A difference between DR systems and the other array generating systems is that DR systems permit for a point to contain more than one symbols. This convention makes the notation of DR systems simpler than the other array generators (cf. [4]). Therefore, we can treat it theoretically.

DR systems will be used as models of the pattern formation of living things. We also think that there are fruitful results which will be obtained from theoretical investigations of DR systems, e.g., a classification of the patterns generated by DR systems and a comparative study of many varieties of DR systems (deterministic, λ-free or propagating, homomorphic image, and so on).

DR systems also relate to process assignment problems for parallel computing. Namely, each symbol of the alphabet corresponds a process and each point in the rectangular array corresponds to a processor which can compute arbitrary number of processes in a time step. Creations and destructions of processes during the computation may be described by the distributed rewriting rules. A process P is assigned to one of the neighbouring processors of the processor which has computed the parent process of P; i.e., if the parent process of P is computed by the processor at (x, y), then P is assigned to the processor at $(x+1, y)$, $(x-1, y)$, $(x, y+1)$, $(x, y-1)$, or (x, y) according to the direction symbol.

DR systems do not contain the issues of communications and synchronizations for parallel computing. This is why some decision problems concerning DR systems are effectively decidable with relatively small computational cost. The study of DR systems will be also useful to build more advanced formal models for parallel computation.

1. Preliminaries

Let Σ be a finite *alphabet*. An element of Σ is called a *symbol*. The set of all strings over Σ, including the empty string 1, is denoted by Σ^*. The *length* of a string s is denoted by $\#(s)$. If V is any subset of Σ, $\#_V(s)$ denotes the number of occurrences of symbols of V in s.

For a set A, $\#(A)$ denotes the cardinality of A and $\mathcal{P}(A)$ denotes the power set of A, i.e., the set of all subsets of A.

We denote by $\mathbf{Z}$ the set of all integers and by $\mathbf{N}$ the set of non-negative integers.

Let w be a string in Σ^* and L be a subset of Σ^*. We denote by $alph(w)$ the set of all and only symbols of Σ which actually appear in w, and by $alph(L)$ the set of all and only symbols appearing in the strings of L.

Definition. A mapping h from Θ^* into Σ^* is said to be a *homomorphism* if it satisfies the following conditions.

i) $h(1) = 1$,

ii) $h(a) \in \Sigma^*$ for every $a \in \Theta$, and

iii) $h(w) = h(a_1)h(a_2)\cdots h(a_n)$ for every $w = a_1a_2\cdots a_n$ where $a_i \in \Theta$ for $i = 1, 2, \ldots, n$.

Unless otherwise stated, we treat in this paper homomorphism $h : \Sigma^* \to \Sigma^*$ and we call such h a homomorphism on Σ^*. In this case we define for every integer n the product h^n as follows:

$$h^0 = \text{ the identity of } \Sigma^*,$$

$$h^1 = h, \text{ and } h^{n+1} = h(h^n).$$

The product is again a homomorphism. We shall use the following notations:

$$h^* = \bigcup_{k \in \mathbf{N}} h^k \text{ and } h^+ = h(h^*).$$

A multivalued mapping τ from Σ^* to Σ^* is said to be a substitution on Σ^* if it is a homomorphism from Σ^* to $\mathcal{P}(\Sigma^*)$. Thus a substitution τ is completely defined by the family of sets $\{\tau(a)|a \in \Sigma\}$ and we have $\tau(1) = 1$.

Definition. Let τ be a substitution on Σ^*. A pair $\langle \Sigma, \tau \rangle$ is called a *0L scheme*. A triple $\langle \Sigma, \tau, w \rangle$ where $\langle \Sigma, \tau \rangle$ is a 0L scheme and w is a string in Σ^* is said to be a *0L system*. Let h be a homomorphism on Σ^*. A pair $\langle \Sigma, h \rangle$ is called a deterministic 0L scheme or a *D0L scheme* for short. A triple $\langle \Sigma, h, w \rangle$ where $\langle \Sigma, h \rangle$ is a D0L scheme and w is a string in Σ^* is called a deterministic 0L system or a *D0L system* for short.

We assume the reader to be familiar with the basic notions and results of D0L systems (see, for example, [3]).

2. Distributed rewriting system and its underlying 0L system

In this section the main notion of this paper is developed. Some notations which are specific to this paper are also given.

Let Σ be a finite alphabet and $\mathbf{Z} \times \mathbf{Z}$ be the 2-dimensional rectangular array. A finite subset D of $\Sigma \times (\Sigma \times \{\uparrow, \downarrow, \leftarrow, \rightarrow, \cdot\})^*$ is said to be a set of *distributed rewriting rules* if for any a in Σ there exists a pair (a, T) in D for some $T \in (\Sigma \times \{\uparrow, \downarrow, \leftarrow, \rightarrow, \cdot\})^*$. We sometimes denote $a \rightarrow T$ if (a, T) is in D. A pair $\langle \Sigma, D \rangle$ is called a *distributed rewriting scheme* or a *DR scheme* for short.

A function C from $\mathbf{Z} \times \mathbf{Z}$ to Σ^* is called a *configuration* over Σ. The set of all configurations over Σ is denoted by $C(\Sigma)$. Let C_1 and C_2 be two configurations. The *multiplication* of C_1 and C_2, denoted by $C_1 \| C_2$, is defined as follows:

$$(C_1 \| C_2)(x, y) = C_1(x, y) C_2(x, y) \text{ for any } (x, y) \in \mathbf{Z} \times \mathbf{Z},$$

where $C_1(x, y) C_2(x, y)$ stands for the concatenation of the strings.

Property 2.1. *$C(\Sigma)$ is a free monoid under the multiplication $\|$. The empty configuration ε, i.e., $\varepsilon(x, y) = 1$ for all (x, y), is the unit element of the monoid.* □

A configuration which takes non-empty string at most one point is called a *point configuration* and is denoted by $[s, x, y]$, where (x, y) is the point and s is the string on the point; i.e., $[s, x, y](x, y) = s$, $s \in \Sigma^+$, and $[s, x, y](x', y') = 1$, for any $(x', y') \neq (x, y)$. The point configuration which takes a symbol on the point is said to be a *single point*.

Let $\delta_\uparrow$, $\delta_\downarrow$, $\delta_\leftarrow$, $\delta_\rightarrow$, and $\delta_\cdot$ be homomorphisms from $(\Sigma \times \{\uparrow, \downarrow, \leftarrow, \rightarrow, \cdot\})^*$ to Σ^* given by,

$$\delta_x((a, y)) = \begin{cases} a, & \text{if } x = y \\ 1, & \text{otherwise} \end{cases}$$

where x and y are in $\{\uparrow, \downarrow, \leftarrow, \rightarrow, \cdot\}$. For example, $\delta_\leftarrow((a, \uparrow)(b, \leftarrow)(a, \leftarrow)) = ba$.

The distributed rewriting rules determines a relation on $C(\Sigma)$ as follows.

Definition. Let $\langle \Sigma, D \rangle$ be a DR scheme and let $[a, x, y]$ be a single point. A configuration C is said to be *directory derived* from $[a, x, y]$ if there exists (a, T) in D and C satisfies:

$$\begin{aligned} C(x, y) &= \delta_\cdot(T), \\ C(x+1, y) &= \delta_\rightarrow(T), \\ C(x-1, y) &= \delta_\leftarrow(T), \\ C(x, y+1) &= \delta_\uparrow(T), \\ C(x, y-1) &= \delta_\downarrow(T), \text{ and} \\ C(x', y') &= 1 \text{ if } |x' - x| + |y' - y| > 1. \end{aligned}$$

We denote C by $d[a, x, y]$.

Remark. For a single point $[a, x, y]$, there are different directly derived configurations which are determined by the different elements (a, T') in D. In other words, the directly deriving relation is a non-deterministic mapping, i.e., it is a function from $C(\Sigma)$ to $\mathcal{P}(C(\Sigma))$. □

Definition. Let $[s, x, y]$ is a point configuration such that $s = s_1 s_2 \cdots s_l$. A directly derived configuration $d[s, x, y]$ from $[s, x, y]$ is given by

$$d[s, x, y] = d[s_1, x, y] \| d[s_2, x, y] \| \cdots \| d[s_l, x, y].$$

Definition. Let $\langle \Sigma, D \rangle$ be a DR scheme. Let C_1 and C_2 be two configurations and let E be an enumeration of all elements of $\mathbf{Z} \times \mathbf{Z}$.

i) C_2 is *simultaneously derived* from C_1 under the enumeration E by D (denoted by $C_1\,{}_D\!\Rightarrow_{sim,E} C_2$) if

$$C_2 = d[s_1, x_1, y_1] \| d[s_2, x_2, y_2] \| \cdots,$$

where $E(i) = (x_i, y_i)$ and $s_i = C_1(x_i, y_i)$.

ii) C_2 is *sequentially derived* from C_1 by D (denoted by $C_1\,{}_D\!\Rightarrow_{seq} C_2$) if for some (x, y)

$$C_1(x, y) = s_1 s_2 \cdots s_l \text{ and } C_2 = d[s_i, x, y] \| C',$$

where $C'(x, y) = s_1 s_2 \cdots s_{i-1} s_{i+1} \cdots s_l$ and $C'(x', y') = C_1(x, y)$ for any $(x', y') \neq (x, y)$.

We note that $\varepsilon\,{}_D\!\Rightarrow_x \varepsilon$ by definition ($x = sim, E$ or seq).

Let ${}_D\!\Rightarrow_x$ be one of the derivation relation defined above ($x = sim, E$ or seq), then the reflective and transitive closure of ${}_D\!\Rightarrow_x$ is denoted by ${}_D\!\Rightarrow_x^*$. We omit D and/or x when D and/or x are understood.

Definition. i) A *distributed rewriting system* (abbreviated as a DR system) P is a triple $P = \langle \Sigma, D, C \rangle$, where $\langle \Sigma, D \rangle$ is a DR scheme and C is a configuration over Σ called the axiom of P.

ii) A sequence of configurations C_0, C_1, ... is said to be the derivation sequence by P under x ($x = sim, E$ or seq) if $C_0 = C$ and $C_i \Rightarrow_x C_{i+1}$ for $i \geq 0$.

A DR scheme $\langle \Sigma, D \rangle$ (or system $\langle \Sigma, D, C \rangle$) is called *deterministic* if D is a function from Σ to $(\Sigma \times \{\uparrow, \downarrow, \leftarrow, \rightarrow, \cdot\})^*$. It is easily seen that for a deterministic DR scheme $\langle \Sigma, D \rangle$ and an enumeration E, the simultaneous derivation relation $\Rightarrow_{sim,E}$ is a function on $C(\Sigma)$.

Next we define an underlying 0L system of a DR system. Let $\langle \Sigma, D \rangle$ be a DR scheme. We construct a substitution τ on Σ^* by setting,

$$\tau(a) = \{w \in \Sigma^* | w = \delta(T), (a, T) \in D\},$$

where δ is a surjection from $(\Sigma \times \{\uparrow, \downarrow, \leftarrow, \rightarrow, \cdot\})^*$ to Σ^* given by

$$\delta((a, x)) = a, \text{ for any } x \in \{\uparrow, \downarrow, \leftarrow, \rightarrow, \cdot\}.$$

Then the pair $\langle \Sigma, \tau \rangle$ is a 0L scheme and is called the *underlying 0L scheme* of $\langle \Sigma, D \rangle$. We note that if a DR scheme is deterministic, then the underlying 0L scheme is deterministic.

Definition. Let $P = \langle \Sigma, D, C \rangle$ be a DR system and E be an enumeration on $\mathbf{Z} \times \mathbf{Z}$. A 0L system $\langle \Sigma, \tau, w \rangle$ is said to be the *underlying 0L system* of P under E and denoted by G_E if $\langle \Sigma, \tau \rangle$ is the underlying 0L scheme of $\langle \Sigma, D \rangle$ and $w = C(x_1, y_1)C(x_2, y_2)\cdots$ where $E(i) = (x_i, y_i)$.

We denote the underlying 0L system by G instead of G_E when E is understood or the enumeration has no effect on the 0L system, e.g., when C is a point configuration.

In the sequel, we only consider the deterministic DR systems and the simultaneous derivation under the triangular enumeration E, i.e., $E(n) = (i, j)$, where

$$n = 2k^2 - 2k + 2 + \begin{cases} j & \text{if } i \geq 0 \text{ and } j > 0 \\ k - i & \text{if } i < 0 \text{ and } j \geq 0 \\ 2k - j & \text{if } i \leq 0 \text{ and } j < 0 \\ 3k + i & \text{if } i > 0 \text{ and } j \leq 0 \end{cases}$$

and $k = |i| + |j|$ (see Figure 1).

We sometimes concentrate our attention on a derivation process of a particular symbol rather than on the whole derivation of the configuration. Let $P = \langle \Sigma, D, C \rangle$ be a deterministic DR system and $C_0 = C$, C_1, ... be the derivation sequence by P. If a symbol s is in $alph(C_i(x, y))$, then we call that the single point $[s, x, y]$ is a part of C_i. A sequence of $n + 1$ single points $[s_0, x_0, y_0]_0$, $[s_1, x_1, y_1]_1$, $\ldots$, $[s_n, x_n, y_n]_n$ is said to be a *derivation process* of $[s, x, y]$ by P if it satisfies:

i) $[s_n, x_n, y_n]_n$ is a part of C_i, i.e., $n = i$, $s_n = s$, and $(x_n, y_n) = (x, y)$.

ii) $[s_0, x_0, y_0]_0$ is a part of C_0.

iii) For all $j > 0$, s_{j+1} occurs in $d[s_j, x_j, y_j]$ at (x_{j+1}, y_{j+1}), i.e.,

$$s_{j+1} \in alph(d[s_j, x_j, y_j]_j(x_{j+1}, y_{j+1})).$$

We denote that $[s, x, y] \to^n [s', x', y']$ if there is a derivation process $[s, x, y] = [s_0, x_0, y_0]_0$, $\ldots$, $[s_n, x_n, y_n]_n = [s', x', y']$.

3. Decision problem for boundedness

A DR system is bounded if the number of points which contain non-empty strings in any configurations derived by it is bounded by a given positive integer. In other words, the derivation of a bounded DR system is simulated in a finite (torus boundary) rectangular array. The main theorem proved in this section (Theorem 3.8) ensures that a DR system is effectively determined whether it is bounded or not. Theorem 3.8 will be clear by characterizing ingeniously the symbols in the alphabet.

Definition. A DR system $P = \langle \Sigma, D, C \rangle$ is said to be *bounded* if there is a positive integer N such that $N > \#(\{(x,y) | C_i(x,y) \neq 1\})$ for any $C \Rightarrow^i C_i$.

First we review some properties of symbols in 0L system which are commonly studied in the theory of 0L systems.

Definition. Let $\langle \Sigma, D \rangle$ be a DR scheme and let $\langle \Sigma, h \rangle$ be its underlying 0L scheme.

i) A symbol $s \in \Sigma$ is called *mortal* if 1 is the descendant of s, i.e., $1 \in h^+(s)$.
ii) A non-mortal symbol is called *vital.*
iii) A vital symbol s is said to be *self-embedding* if s appears in some descendant of s, i.e., $usv \in h^+(s)$ for some $uv \in \Sigma^*$. We denote by S the set of all self-embedding symbols.

Next we define bounded and unbounded symbols.

Definition. Let $\langle \Sigma, D \rangle$ be a DR scheme. A symbol $s \in \Sigma$ is said to be *bounded* if the DR system $\langle \Sigma, D, [s, 0, 0] \rangle$ is bounded. A symbol which is not bounded is called *unbounded.*

Property 3.1. *A symbol is unbounded if one of its descendant is unbounded.* □

Since there is no interaction among the symbols in DR system, it is obvious that a DR system $P = \langle \Sigma, D, C \rangle$ is bounded if and only if all symbols appearing in C is bounded. Therefore the subject of this section is to determine whether a symbol is bounded or not. It is trivial that any mortal symbol is bounded. And from Property 3.1 it is obvious that a vital non self-embedding symbol is bounded if and only if all its self-embedding descendants are bounded. Then our task is reduced to decide whether or not a self-embedding symbol is bounded.

Definition. Let s be a self-embedding symbol. The symbol s is said to be *single directed* if for any $[s, 0, 0] \Rightarrow^* C$ there exists a pair of integers (p, q) such that $s \in alph(C(x,y))$ implies $(x,y) = (p,q)$. A self-embedding symbol which is not single directed is called *multiple directed.*

A multiple directed symbol is easily proved to be unbounded (Property 3.2 below). But it is not so easy to decide whether a single directed symbol is bounded or not. For example, consider the following example.

Example. Let $P_1 = \langle \{a\}, \{a \to (a, \cdot)(a, \rightarrow)\}, [a, 0, 0] \rangle$, $P_2 = \langle \{a, b\}, \{a \to (a, \rightarrow)(b, \rightarrow), b \to (b, \uparrow)\}, [a, 0, 0] \rangle$, and $P_3 = \langle \{a, b\}, \{a \to (a, \rightarrow)(b, \rightarrow), b \to (b, \rightarrow)\}, [a, 0, 0] \rangle$ be DR systems. Then a in P_1 is multiple directed and hence P_1 is unbounded. On the other hand, a in P_2 and P_3 is single directed, but P_2 is unbounded, while P_3 is bounded. The first few derivation steps are illustrated in Figure 2. □

Property 3.2. *A multiple directed symbol is unbounded.*

Proof. Let s be a multiple directed symbol. There is a configuration C such that $[s, 0, 0] \Rightarrow^m C$ and $s \in alph(C(p_0, q_0)) \cap alph(C(p_1, q_1))$ where $(p_0, q_0) \neq (p_1, q_1)$. Then for any positive integer k, s occurs at the $k+1$ points (kp_0, kq_0), $((k-1)p_0 + p_1, (k-1)q_0 + q_1)$, ..., and (kp_1, kq_1) in the configuration C' such that $[s, 0, 0] \Rightarrow^{km} C'$. □

The next lemma makes an essential property of single directed symbols clear.

Lemma 3.3. *Let $P = \langle \Sigma, D, [s, 0, 0] \rangle$ be a DR system where $s \in \Sigma$ is self-embedding. Let $C_0 = [s, 0, 0]$, C_1, ... be the derivation sequence by P. If the symbol s is single directed, then for any i, j, (x, y), and (x', y') such that $s \in alph(C_i(x, y))$ and $s \in alph(C_j(x', y'))$ we have $(x/i, y/i) = (x'/j, y'/j)$.*

Proof. Suppose that there are two points $[s, x, y]$ and $[s, x', y']$ which are parts of C_i and C_j, respectively, such that $(x/i, y/i) \neq (x'/j, y'/j)$. If $i = j$, then s is not single-directed. Otherwise, two different single points $[s, jx, jy]_{ij}$ and $[s, ix', iy']_{ij}$ are parts of C_{ij}, that is, s is not single directed. □

The vector whose uniqueness is proved in the above lemma is called the *unit derivation vector* of s. Then the next lemma gives the necessary and sufficient condition for a single directed symbol to be bounded.

Lemma 3.4. *A single directed symbol s is bounded if and only if for any symbol $a \in S$ which is a descendant of s is single directed and the unit derivation vector of a is that of s.*

Proof. Only if part: If a self-embedding symbol s is bounded, then s and all its self-embedding descendants are single directed by Properties 3.1 and 3.2. Let (x_s, y_s) be the unit derivation vector of s and $[s, 0, 0]$, ..., $[s, ix_s, iy_s]_i$ be one of the derivation processes of s. Now assume that a descendant $a \in S$ of s has a unit derivation vector (x_a, y_a) such that $(x_a, y_a) \neq (x_s, y_s)$. There are derivation processes $[s, 0, 0]$, $\ldots$, $[a, x, y]_m$ and $[a, 0, 0]$, $\ldots$, $[a, jx_a, jy_a]_j$. Then, for any positive integer n, the configuration C_{m+nM} contains a at n points:

$$pM(x_s, y_s) + (n - p)M(x_a, y_a) + (x, y) \text{ for } 0 \leq p < n,$$

where M is the least common multiple of i and j. This implies that s is unbounded, and hence we have $(x_a, y_a) = (x_s, y_s)$.

If part: First assume that a descendant t of s is not self-embedding. If t is in $alph(C(x, y))$ for some configuration C which satisfies $[s, 0, 0] \Rightarrow^n C$, in other words, t has a derivation process $T = \{[s, 0, 0], \ldots, [t, x, y]_n\}$; then there is a final segment $[s, x_{n-j}, y_{n-j}]_{n-j}$, $\ldots$, $[t, x, y]_n$ of T such that $j \leq \#(\Sigma)$ because t is not self-embedding. Thus we have $|x - nx_s| + |y - ny_s| \leq \#(\Sigma)$ because t is self-embedding. Next assume that $a \in S$ is a descendant of s and $A = \{[s, 0, 0], \ldots, [a, u, v]_n\}$ be a derivation process of a. Then there is an initial segment $[s, 0, 0]$, $\ldots$, $[a, u_0, v_0]_j$ of A such that $j \leq \#(\Sigma)$ and $a_m \neq a$ for $0 < m < j$. Since the unit derivation vector of s equals to that of a, we have $(u, v) - (u_0, v_0) = (n - j)(x_s, y_s)$. Therefore we have $|u - nx_s| + |v - ny_s| < |u_0| + |v_0| + |jx_s| + |jy_s|$, where the right side hand is a constant which is dependent only on the distributed rewriting rules. Thus, for any integer n and the configuration C which satisfies $[s, 0, 0] \Rightarrow^n C$, the cardinarity of the set $\{(x, y) | C(x, y) \neq 1\}$ is bounded. □

Now the remaining task is to determine whether a self-embedding symbol is single directed or not. This is done as follows: Let $\langle \Sigma, D \rangle$ be a DR scheme and let s be a self-embedding symbol. We construct a non-deterministic finite automaton $M_s = \langle \Sigma, \{\uparrow, \downarrow, \leftarrow, \rightarrow, \cdot\}, \delta, s, F \rangle$ where $F = \{s\}$ and

$$\delta(b, d) = \{c \in \Sigma | b \to x(c, d)y \text{ is in } D \text{ for some } xy \in (\Sigma \times \{\uparrow, \downarrow, \leftarrow, \rightarrow, \cdot\})^*\}.$$

Then M_s satisfies the following property.

Property 3.5. *For any derivation process $[s, 0, 0]_0$, $\ldots$ $[s, x_n, y_n]_n$, there exists a string $w = d_1 \cdots d_n \in \{\uparrow, \downarrow, \leftarrow, \rightarrow, \cdot\}^*$ in $L(M_s)$ such that d_i is the direction from the point $[s_{i-1}, x_{i-1}, y_{i-1}]_{i-1}$ to the point $[s_i, x_i, y_i]_i$ for $1 \leq i \leq n$.* □

Let $\mathcal{E}_s$ be the regular expression such that $L(\mathcal{E}_s) = L(M_s)$. We can assume that $\mathcal{E}_s$ contains no + operator, for otherwise we add $\mathcal{E}\mathcal{E}_1\mathcal{E}'$ and $\mathcal{E}\mathcal{E}_2\mathcal{E}'$ to $\mathcal{E}_s$ if $\mathcal{E}(\mathcal{E}_1 + \mathcal{E}_2)\mathcal{E}'$

is in $\mathcal{E}_s$ and we add $\mathcal{E}(\mathcal{E}_1^*\mathcal{E}_2^*)^*\mathcal{E}'$ to $\mathcal{E}_s$ if $\mathcal{E}(\mathcal{E}_1+\mathcal{E}_2)^*\mathcal{E}'$ is in $\mathcal{E}_s$. We inductively define a function f from $\mathcal{E}_s - \{\emptyset, \varepsilon\}$ to $(\mathbf{N}\times\mathbf{N})\times\mathbf{N}$ as follows:

i) if $\mathcal{E} = d \in \{\uparrow, \downarrow, \leftarrow, \rightarrow, \cdot\}$, then $f(\mathcal{E}) = (\mathbf{v}, 1)$ where

$$\begin{aligned} \mathbf{v} &= (1,0) \text{ if } d = \rightarrow, \\ \mathbf{v} &= (-1,0) \text{ if } d = \leftarrow, \\ \mathbf{v} &= (0,1) \text{ if } d = \uparrow, \\ \mathbf{v} &= (0,-1) \text{ if } d = \downarrow, \text{ and} \\ \mathbf{v} &= (0,0) \text{ if } d = \cdot. \end{aligned}$$

ii) if $\mathcal{E} = \mathcal{E}_1\mathcal{E}_2$, $f(\mathcal{E}_1) = (\mathbf{v_1}, n_1)$, and $f(\mathcal{E}_2) = (\mathbf{v_2}, n_2)$, then $f(\mathcal{E}) = (\mathbf{v_1}+\mathbf{v_2}, n_1+n_2)$.

iii) if $\mathcal{E} = \mathcal{E}_1^*$ and $f(\mathcal{E}_1) = (\mathbf{v_1}, n_1)$, then $f(\mathcal{E}) = (k\mathbf{v_1}, kn_1)$ where k is a variable which does not appear in n_1.

If $f(\mathcal{E}) = ((x,y),n)$ is in $\mathcal{E}_s$ and n, x, and y have variables $k_1, \ldots, k_l$, then it is easily seen that for any non-negative integers $c_1, \ldots, c_l$

$$[s,0,0] \rightarrow^{n(c_1,\ldots,c_l)} [s, x(c_1,\ldots,c_l), y(c_1,\ldots,c_l)]$$

where $n(c_1,\ldots,c_l)$, $x(c_1,\ldots,c_l)$, and $y(c_1,\ldots,c_l)$ are the integers which are obtained by the assignment $c_1, \ldots, c_l$ for $k_1, \ldots, k_l$ from the definition of f and Property 3.5.

Then the next lemma ensures that a symbol is effectively determined whether it is single directed or not.

Lemma 3.6. *The following two conditions are equivalent.*

i) A self-embedding symbol s is single directed.

ii) For any formulas $\mathcal{E}$ and $\mathcal{E}'$ in $\mathcal{E}_s$ such that $f(\mathcal{E}) = (\mathbf{v}, n)$ and $f(\mathcal{E}') = (\mathbf{v}', n')$ we have $\mathbf{v}/n = \mathbf{v}'/n'$ and $\mathbf{v}/n$ does not contain any variable.

Proof. i) $\rightarrow$ ii). The proof immediately follows from Lemma 3.3.

ii) $\rightarrow$ i). Let $C_0 = [s,0,0], \ldots, C_n$ be the derivation sequence by the DR system $\langle \Sigma, D, [s,0,0] \rangle$. If s is in $alph(C_n(x,y))$ and $alph(C_n(x',y'))$, then by the condition ii) we have $(x,y) = (x',y')$. This implies that s is single directed. □

Proposition 3.7. *A symbol is effectively determined whether it is bounded or not.*

Proof. This is an immediate consequence of Lemma 3.4 and Lemma 3.6. □

Now the following theorem is proved.

Theorem 3.8. *A DR system is effectively determined whether or not it is bounded.*
□

4. Decision problem for explosion and flatten theorem

In this section, we concentrate our attention on the length of the string in each point; namely, for a given DR system $P = \langle \Sigma, D, C \rangle$, we ask whether there exists an integer N such that $N > \#(C_i(x,y))$ for any $C \Rightarrow^i C_i$ and any (x,y). If there is no such N, we say that P explodes. We show that a DR system is effectively determined whether it explodes or not (Theorem 4.7). Then we prove that certain DR system is made "flat"; in other words, for a DR system P which explodes, there is a DR system P' which does not explode and has the same underlying 0L system with P (Theorem 4.8). We conclude this section with a characterization of the DR systems by combining the notions of boundedness and explosion.

First we characterize the symbols of a DR system by their growth order which is defined by the growth function of the underlying 0L system.

Let $\langle \Sigma, D \rangle$ be a DR scheme and let $\langle \Sigma, h \rangle$ be its underlying 0L scheme. For a symbol s in Σ, let $f_s(n)$ be the growth function of the 0L system $\langle \Sigma, h, s \rangle$, i.e., $f_s(n) = \#(h^n(s))$. We define the *growth order* (or *order* for short) of the symbol s as follows:

Polynomial $k(\geq 0)$: If for any c there exists N such that $f_s(n) > cn^{k-1}$ for any $n > N$ and there exists a constant c such that for any $n \geq 0$ $f_s(n) \leq cn^k$.

Exponential: If for any k there is N such that $f_s(n) > n^k$ for any $n > N$.

Null: If there exists N such that $f_s(n) = 0$ for any $n > N$.

We denote the order of a symbol s by $order(s)$ as follows:

$$order(s) = \begin{cases} \infty, & \text{if } s \text{ is exponential,} \\ k, & \text{if } s \text{ is polynomial } k, \\ -\infty, & \text{if } s \text{ is null.} \end{cases}$$

The following property is obvious from the definition of the order and the standard properties of homomorphisms.

Property 4.1. *Let $\langle \Sigma, D \rangle$ be a DR scheme and $\langle \Sigma, h \rangle$ be the underlying 0L scheme of it.*

i) A symbol s is null if and only if s is mortal.

ii) For any symbol a, $order(a) = \mathrm{Max}_{b \in alph(h(a))} order(b)$.

iii) For any non negative order symbol a (i.e., a is vital), there is a self-embedding symbol s in $alph(h^+(a))$ such that $order(a) = order(s)$.

iv) If a symbol s is multiple directed, then order$(s) = \infty$. □

Property 4.1 says that the null symbols are effectively determined. Since the growth function of a D0L system is easily calculated, the order of any symbol is also effectively determined. The well established method to calculate the growth function of a D0L system utilizes the growth matrix, i.e., the matrix $M = [m_{ij}]$ where $m_{ij} = \#_{\{a_j\}}(h(a_i))$ and $\Sigma = \{a_1, a_2, \ldots, a_k\}$. Then $f_{a_i}(n) = \pi M^n \eta$ where

$$\pi = \begin{array}{c} \begin{array}{ccccccc} & & & i & & & \end{array} \\ \begin{pmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{pmatrix} \end{array}$$

and $\eta = (1, \ldots, 1)^T$.

The next lemma clarifies a subtle property of a polynomial order symbol, which will be useful in the sequel.

Lemma 4.2. *If the order of a self-embedding symbol s is polynomial i, then:*

i) There are exactly one self-embedding symbol of order i and no symbol which are order more than i contained in $h^n(s)$ for any $n > 0$.

ii) There is at least one self-embedding symbol of order $i-1$ contained in some $h^m(s)$ where m is a positive integer less than $\#(\Sigma)$.

Proof. i) First observe that all symbols in $alph(h^+(s))$ are order less than or equal to i and that there exists at least one symbol of order i contained in $h^n(s)$ for any $n > 0$ by Property 4.1 ii) and iii). Then assume there are two symbols a and b of order i contained in $h^m(s)$ for some $m > 0$, i.e., $uavbw = h^m(s)$ for some $uvw \in \Sigma^*$. If $s \in alph(h^+(a))$ and $s \in alph(h^+(b))$, then s would be exponential. Therefore we can assume, without loss of generality, that $s \notin alph(h^k(b))$ and $s \in h^k(s)$ for some $k > 0$. Let $f_s(x)$ and $f_b(x)$ be the growth functions of s and b. Then we have

$$f_s(kx) > \sum_{n=0}^{x} f_b(kn - m)$$

for any $x > 0$. Since b is order i, $f_b(x) > cx^{i-1}$ for any sufficiently large x, and hence for any c there exists N such that $f_s(x) > cx^i$ for all $x > N$. This contradict that s is order i. Therefore there is exactly one order i symbol contained in $h^n(s)$ for any $n > 0$.

ii) If there was no order $i-1$ symbol contained in any $h^m(s)$, then we should have

$$f_s(x) \leq \sum_{n=0}^{x} (cn^{i-2} + O(i-3))$$

where $O(i-3)$ stands for the terms of order $i-3$ and this should imply that s is order $i-1$. Let a be a self-embedding symbol of order $i-1$ and $[s,0,0]$, ..., $[a_i,x_i,y_i]_i$, ..., $[a,x,y]_m$ be a derivation process of a. If $m > \#(\Sigma)$, then there are integers i and j such that $i \neq j$ and $a_i = a_j$. Therefore we have that a_i is self-embedding and $order(a_i) = i-1$. Then the proof is completed. □

Lemma 4.2 says that for a self-embedding symbol s whose order is i, there exist $i+1$ self embedding symbols s_i, s_{i-1}, $\ldots, s_0$ such that $s_i = s$, $s_{j-1} \in alph(h^+(s_j))$, and $order(s_j) = order(s_{j+1}) - 1 = j$. The sequence $\{s_i, s_{i-1}, \ldots, s_0\}$ is called a *descendant sequence* of s.

This lemma also proves that for a self embedding symbol s of polynomial order, there is only one derivation process $[s,0,0]$, ..., $[s,x,y]_c$ which satisfies the following conditions:

i) $c \leq \#(\Sigma)$.

ii) $s_j \neq s$ for $0 < j < c$.

In this case, c is called the *length* of cycle of s. (Of course, the unit derivation vector is $(x/c, y/c)$.)

Now we define the notion of explosion.

Definition. Let $P = \langle \Sigma, D, C \rangle$ be a DR system and $C_0 = C, C_1, \ldots$ be the derivation sequence. If for any integer N there is a positive integer M such that for any $n \geq M$, $\#(C_n(x,y)) > N$ for some (x,y), then P is said to be *explosive* or P is said to *explode*. A DR system which is not explosive is called *flat*.

The next property directly follows from the definition and the properties of the growth order.

Property 4.3. *Let $P = \langle \Sigma, D, C \rangle$ be a DR system. P is explosive if*

i) There is a exponential symbol contained in $C(x,y)$ for some (x,y).

ii) There is a symbol s contained in $C(x,y)$ for some (x,y) such that $order(s) > 2$.

iii) P is bounded and there is a symbol s contained in $C(x,y)$ for some (x,y) such that $order(s) \geq 1$. □

If P is unbounded and for any (x,y) all symbols in $C(x,y)$ have order less than 3, then there are two possibilities: P explodes or P does not explode. Indeed, consider the following examples.

Example. Let $P_1 = \langle \{a,b,c\}, D_1, [a,0,0] \rangle$ be a DR system, where $D_1 = \{a \to (a,\cdot)(b,\to)(c,\cdot), b \to (b,\to), c \to (c,\cdot)\}$. Then $order(a) = 1$ and $order(b) = order(c) = 0$. As seen in Figure 3, P_1 is explosive.

Next let $P_2 = \langle\{a,b,c\}, D_2, [a,0,0]\rangle$ be a DR system, where $D_2 = \{a \to (a,\cdot)(b,\uparrow), b \to (b,\uparrow)(c,\to), c \to (c,\to)\}$. Then $order(a) = 2$, $order(b) = 1$, and $order(c) = 0$. But P_2 does not explode (see Figure 3). □

The following lemmas give the necessary and sufficient conditions for a polynomial DR system to explode.

Lemma 4.4. *Let $P = \langle\Sigma, D, C\rangle$ be a DR system whose underlying 0L system $G = \langle\Sigma, h, w\rangle$ is polynomial order more than 0. Then P explodes if there exist self-embedding symbols s and t in $alph(h^+(w))$ such that $\{s,t\}$ is the descendant sequence of s and the unit derivation vector of s equals to that of t.*

Proof. Let L be the least common multiple of the length of cycles of s and t and (x_s, y_s) be the unit derivation vector of s (and hence of t). Since s and t occur in some configurations, we can assume that $C_i(x,y) = usv$, $C_j(x',y') = u'tv'$, and $[s,x,y] \to^{j-i} [t,x',y']$. Then, for any n, the number of the occurrences of t in $C_{j+Ln}(x' + Lnx_s, y' + Lny_s)$ is more than n. □

Lemma 4.5. *With the same assumptions as in Lemma 4.4, P explodes if there exist self-embedding symbols r, s, and t in $alph(h^+(w))$ such that $\{r,s,t\}$ is the descendant sequence of r and for the unit derivation vectors $\mathbf{u_r} = (x_r, y_r)$, $\mathbf{u_s} = (x_s, y_s)$, and $\mathbf{u_t} = (x_t, y_t)$ of r, s, and t satisfy,*

$$x_r y_s + x_s y_t + x_t y_r - x_r y_t - x_t y_s - x_s y_r = 0.$$

Proof. Let c_r, c_s, ant c_t be the length of the cycles of r, s, and t, respectively. We use the vector notation $C_i(\mathbf{x})$ instead of $C_i(x,y)$ where $x = (x,y)$. Since r, s, and t occur in some configurations we have:

$$r \in alph(c_i(\mathbf{X})), s \in alph(C_{i+i'}(\mathbf{X} + \mathbf{X}')), \text{ and } t \in alph(C_{i+i'+i''}(\mathbf{X} + \mathbf{X}' + \mathbf{X}''))$$

for some positive constant integers i, i', and i'' and constant vectors $\mathbf{X}$, $\mathbf{X}'$, and $\mathbf{X}''$. Then t typically occurs in

$$C_{i+i'+i''+kc_r+lc_s+mc_t}(\mathbf{X} + \mathbf{X}' + \mathbf{X}'' + kc_r\mathbf{u_r} + lc_s\mathbf{u_s} + mc_t\mathbf{u_t})$$

for some k,l, and $m \geq 0$. Now we must show that for any $N \geq 0$ there exist N non-negative integer solution (k,l,m) of the equations

$$\begin{cases} kc_r + lc_s + mc_t = L \\ kc_r x_r + lc_s x_s + mc_t x_t = X \\ kc_r y_r + lc_s y_s + mc_t y_t = Y \end{cases}$$

for some appropriate constants L, X, and Y. Indeed this holds since the determinant of the coefficients matrix satisfies

$$\det \begin{vmatrix} c_r & c_s & c_t \\ c_r x_r & c_s x_s & c_t x_t \\ c_r y_r & c_s y_s & c_t y_t \end{vmatrix} = 0$$

because of the assumption of this lemma. □

Next we prove the reverse of Lemma 4.4 and Lemma 4.5.

Lemma 4.6. *Let $P = \langle \Sigma, D, C \rangle$ be a DR system whose underlying 0L system $G = \langle \Sigma, h, w \rangle$ is polynomial order less than 3. If P explodes, then*

i) there exist self-embedding symbols s and t in $alph(h^+(w))$ such that $\{s, t\}$ is the descendant sequence of s and the unit derivation vector of s equals to that of t.

ii) there exist self-embedding symbols r, s, and t in $alph(h^+(w))$ such that $\{r, s, t\}$ is the descendant sequence of r and for the unit derivation vectors $\mathbf{u_r} = (x_r, y_r)$, $\mathbf{u_s} = (x_s, y_s)$, and $\mathbf{u_t} = (x_t, y_t)$ of r, s, and t, we have

$$x_r y_s + x_s y_t + x_t y_r - x_r y_t - x_t y_s - x_s y_r = 0.$$

Proof. Since P explodes and $\#(\Sigma)$ is finite, for any $N > 0$, there are a configuration C_i derived from C and a self-embedding symbol t such that $\#_{\{t\}}(C_i(x)) > N$. From Lemma 4.2, there are other self-embedding symbols s_0, s_1, … such that $\{s_0, s_1, \ldots, t\}$ is the descendant sequence of s_0, s_0 occurs in $C_j(\mathbf{x_0})$ for some $j < \#(\Sigma)$, and that $[s_0, \mathbf{x_0}]$ derives αN occurances of t in $C_i(\mathbf{x})$ for some constant $0 < \alpha \leq 1$. Since the order of G is less than 3, we must consider the following two cases:

Case 1. The descendant sequence is $\{s, t\}$. Letting $\mathbf{u_s} = (x_s, y_s)$ and $\mathbf{u_t} = (x_t, y_t)$ be the unit derivation vectors and c_s and c_t be the length of cycles of s and t, respectively, we have that the equations

$$\begin{cases} lc_s + mc_t = i - j - j' \\ lc_s \mathbf{u_s} + mc_t \mathbf{u_t} = \mathbf{x} - \mathbf{x_0} - \mathbf{x'} \end{cases}$$

have αN solutions, where j' is a non-negative constant and $\mathbf{x'}$ is a constant vector. The necessary and safficient condition for this is

$$\operatorname{rank} \begin{pmatrix} 1 & 1 \\ x_s & x_t \\ y_s & y_t \end{pmatrix} = 1,$$

and hence we have that $x_s = x_t$ and $y_s = y_t$.

Case 2. The descendant sequence is $\{r, s, t\}$. Let $\mathbf{u_r} = (x_r, y_r)$, $\mathbf{u_s} = (x_s, y_s)$, and $\mathbf{u_t} = (x_t, y_t)$ be the unit derivation vectors and c_r, c_s, and c_t be the length of cycles of r, s, and t, respectively. We have that the equations

$$\begin{cases} kc_r + lc_s + mc_t = i - j - j' - j'' \\ kc_r\mathbf{u_r} + lc_s\mathbf{u_s} + mc_t\mathbf{u_t} = \mathbf{x} - \mathbf{x_0} - \mathbf{x}' - \mathbf{x}'' \end{cases}$$

have αN solutions, where j' and j'' are non-negative constants and $\mathbf{x}'$ and $\mathbf{x}''$ are constant vectors. The necessary and safficient condition for this is

$$\operatorname{rank}\begin{pmatrix} 1 & 1 & 1 \\ x_r & x_s & x_t \\ y_r & y_s & y_t \end{pmatrix} \leq 2.$$

and hence we have $x_r y_s + x_s y_t + x_t y_r - x_r y_t - x_t y_s - x_s y_r = 0$. □

Now the first main result of this section becomes obvious.

Theorem 4.7. *A DR system is effectively determined whether or not it explodes.* □

Then we state and prove the second main theorem of this section.

Theorem 4.8. *Let $P = \langle \Sigma, D, C\rangle$ be a DR system and $G = \langle \Sigma, h, w\rangle$ be its underlying 0L system. If the orders of all symbols in alph(w) are less than 3, then there is a DR system $P' = \langle \Sigma, D', C\rangle$ such that the underlying 0L system of P' is G and that P' does not explode.*

Proof. We construct D' as follows:

i) For any $a \in \Sigma$, if $h(a) = s_1 \cdots s_l$, then

$$a \rightarrow (s_1, x_1) \cdots (s_l, x_l)$$

is in D' where $x_i = \uparrow$ if s_i is order 0 or $-\infty$, $x_i = \rightarrow$ if s_i is order 1, and $x_i = \cdot$ if s_i is order 2.

ii) D' contains no other rewriting rules.

Then G is the underlying 0L system of the DR system $P' = \langle \Sigma, D', C\rangle$, and P' does not explode because of Lemmas 4.4, 4.5, and 4.6. □

Finally we summarize in Table 1 the relationship among the notions of growth order, boundedness, and explosion.

maximum order of symbols in the initial configuration	bounded	unbounded
exponential	explosive	
higher order (> 2)	explosive	
1 or 2	explosive	flat or explosive (flatten by modifying rewriting rules)
0 or null	flat	non exist

Table 1

Acknowledgments

The author is greatly indebted to Mr. Naouki Horai for his excellent suggestion which makes the proof of Lemma 3.6 complete. The author is also very grateful Dr. Yohji Kohda, Mr. Hideki Yamanaka, and the other members of the Institute for their valuable suggestions and discussions.

References

[1] Milgram, D. & Rosenfeld, A. (1972), Array automata and array grammars, Proc. IFIP Congress 71, North-Holland, Amsterdam, 69-74.

[2] Pfaltz, J. L. & Rosenfeld, A. (1969), Web grammars, Proc. 1st Intl. Joint Conf. on Artificial Intelligence, 609-619.

[3] Rozenberg, G. & Salomaa, A. (1980), The mathematical theory of L systems, Academic Press, New York.

[4] Siromoney, G., Siromoney, R., & Krithivasan, K. (1973), Picture languages with array rewriting rules, Information and Control 22, 447-470.

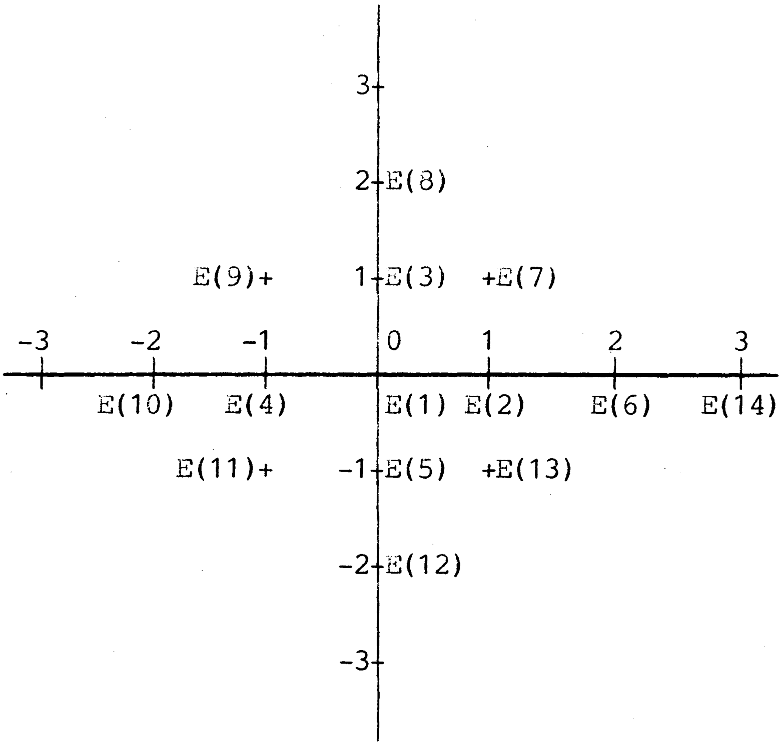

Figure 1

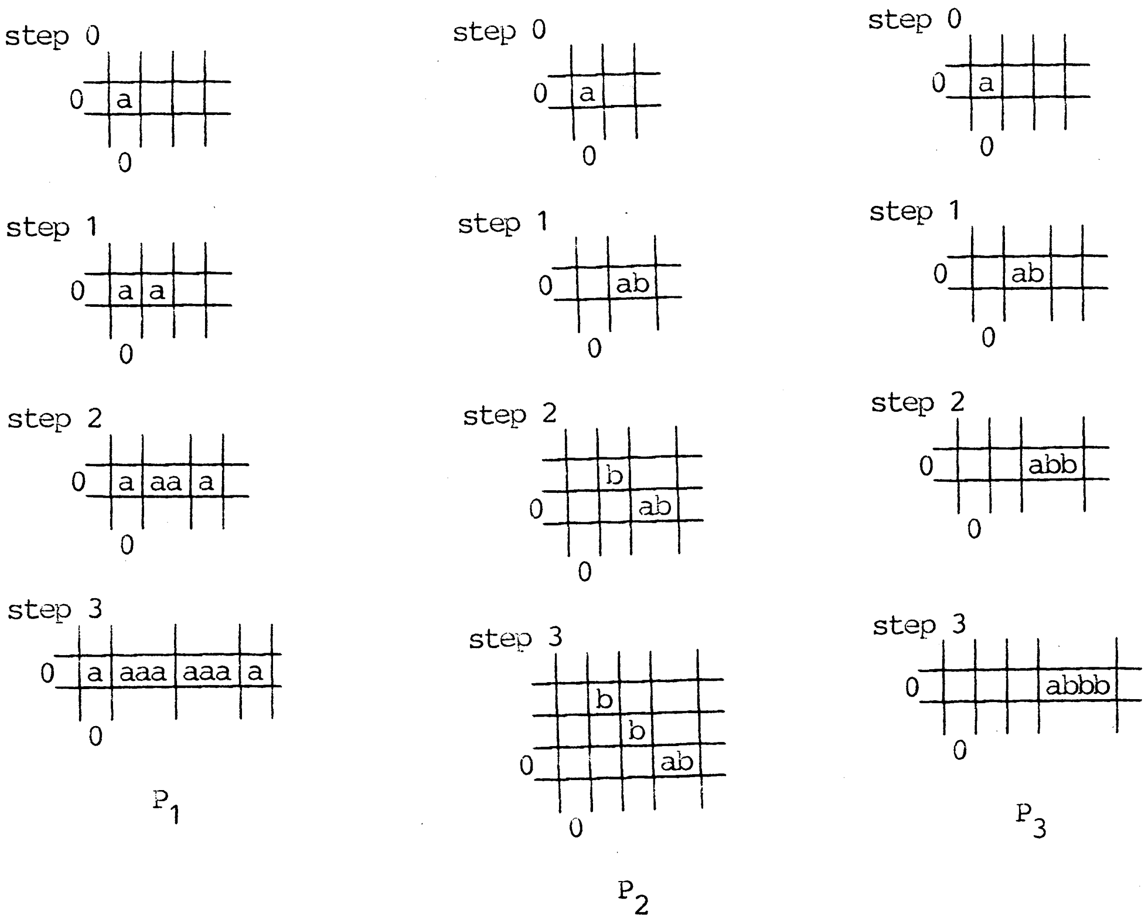

Figure 2

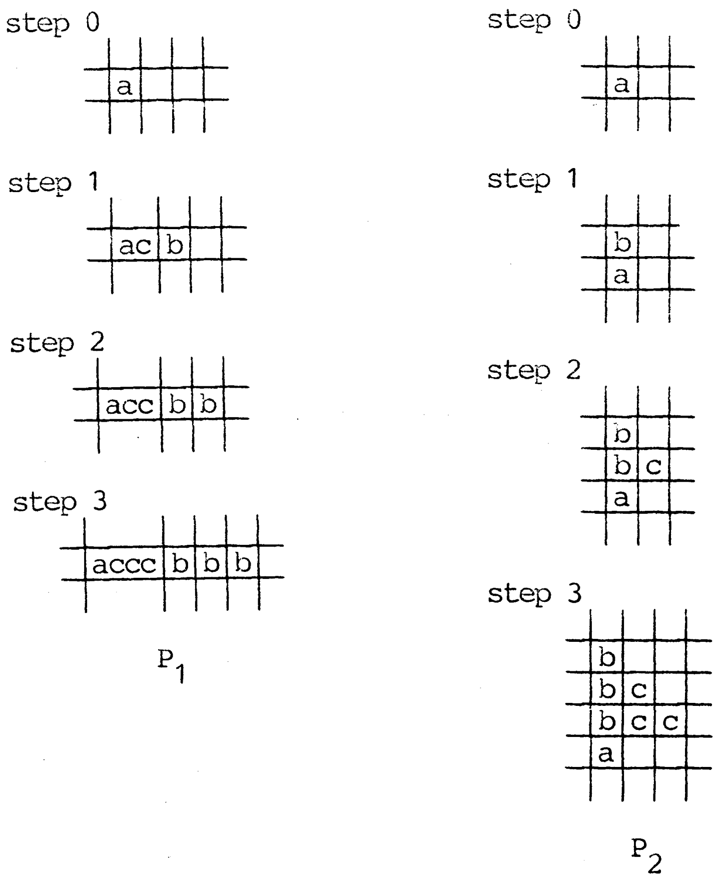
step 0
a
step 1
ac
b
step 2
acc
b
b
step 3
accc
b
b
b
P_1
step 0
a
step 1
b
a
step 2
b
b
c
a
step 3
b
b
c
b
c
c
a
P_2

Figure 3